The two-colour Rado number for the equation $ax + by = (a + b)z$

Running Title: **Rado number for** $ax + by = (a + b)z$

Swati Gupta[*†] **J. Thulasirangan**[‡] **Amitabha Tripathi**[§¶]

Abstract

For relatively prime positive integers a and b, let $n = \mathcal{R}(a, b)$ denote the least positive integer such that every 2-colouring of $[1, n]$ admits a monochromatic solution to $ax + by = (a + b)z$ with x, y, z distinct integers. It is known that $\mathcal{R}(a, b) \leq 4(a + b) + 1$. We show that $\mathcal{R}(a, b) = 4(a + b) + 1$, except when $(a, b) = (3, 4)$ or $(a, b) = (1, 4k)$ for some $k \geq 1$, and $\mathcal{R}(a, b) = 4(a + b) - 1$ in these exceptional cases.

Keywords: Schur numbers; Rado numbers; colourings; monochromatic solution; regular equation
MSC 2010: 05C55, 05D10

1 Introduction

Motivated by a desire to prove Fermat's Last Theorem, in 1916 Schur [22] proved his celebrated result that bears his name and states that for every positive integer r, there exists a least positive integer $s = \mathtt{s}(r)$ such that for every r-colouring of the integers in the interval $[1, s]$, there exists a monochromatic solution to the equation $x + y = z$. The only exact values of $\mathtt{s}(r)$ that are known are $\mathtt{s}(1) = 2$, $\mathtt{s}(2) = 5$, $\mathtt{s}(3) = 14$, and $\mathtt{s}(4) = 45$. Schur's Theorem was generalized in a series of results in the 1930's by Rado [17, 18, 19] leading to a complete resolution to the following problem: characterize systems of linear homogeneous equations with integral coefficients $\mathcal{L}$ such that for a given positive integer r, there exists a least positive integer $n = \mathcal{R}(\mathcal{L}; r)$ such that every r-colouring of the integers in the interval $[1, n]$ yields a monochromatic solution to the system $\mathcal{L}$. Rado's result is particularly easily stated when $\mathcal{L}$ consists of a single equation. In this case, $\mathcal{R}(\mathcal{L}; r)$ exists for every r precisely when some nonempty subset of the coefficients of the single linear equation $\mathcal{L}$ sum to 0, and we say that $\mathcal{L}$ is `regular`. Rado also proved that $\mathcal{R}(\mathcal{L}; 2)$ exists for the single linear homogeneous

[*]Operations Research Center, MIT, Cambridge, MA 02139
[†]Work done while at Department of Computer Science & Engineering, Indian Institute of Technology, Hauz Khas, New Delhi
`e-mail:swatig@mit.edu`
[‡]Work done while at Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi
`e-mail:thulasirangan@gmail.com`
[§]Department of Mathematics, Indian Institute of Technology, Hauz Khas, New Delhi – 110016
`e-mail:atripath@maths.iitd.ac.in`
[¶]*Corresponding author*

equation if and only if there are *at least* three nonzero coefficients and both positive and negative coefficients.

There has been a growing interest in the determination of the Rado numbers $\mathcal{R}(\mathcal{L}; r)$, particularly when $\mathcal{L}$ is a single equation and $r = 2$; for instance, see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 15, 20, 21]. In 1982 Beutelspacher & Brestovansky [1] proved that the 2-colour Rado number for the equation $x_1 + \cdots + x_{m-1} - x_m = 0$ equals $m^2 - m - 1$ for each $m \geq 3$. Jones & Schaal [13] generalized this by considering the equation $a_1x_1 + \cdots + a_{m-1}x_{m-1} - x_m = 0$ for positive integers $a_1, \ldots, a_{m-1}$ and resolving the problem when $\min\{a_1, \ldots, a_{m-1}\} = 1$. Hopkins & Schaal [12] resolved the problem in the case $\min\{a_1, \ldots, a_{m-1}\} = 2$ and gave bounds in the general case which they conjectured to hold, and this was proved by Guo & Sun in [7]. They showed that the 2-colour Rado number for the general case equals $as^2 + s - a$, where $a = \min\{a_1, \ldots, a_{m-1}\}$ and $s = a_1 + \cdots + a_{m-1}$. Kosek & Schaal [15] determined the 2-colour Rado number for the non-homogeneous equation $x_1 + \cdots + x_{m-1} - x_m = c$ for several ranges of values of c.

Suppose $\mathcal{L}$ represents the linear equation $a_1x_1 + \ldots + a_mx_m = 0$. We may assume $m \geq 3$. In view of Rado's result on single linear homogeneous regular equations, the only equations to consider when $m = 3$ are when $a_1 + a_2 = 0$ or when $a_1 + a_2 + a_3 = 0$. The first case was completely resolved by Harborth & Maasberg [10, 11] for the 2-colour case, but the cases when $r \geq 3$ remain open. In the second case, Burr & Loo [4] provided the upper bound $4(a_1 + a_2) + 1$ for the 2-colour case, and Landman & Robertson [16] showed this to be the correct value when $a_1 = 1$ and $4 \nmid a_2$. We completely resolve the problem for the 2-colour case. We prove:

Theorem. *Let a_1, a_2 be relatively prime positive integers. Then*

$$\mathcal{R}\Big(a_1x_1 + a_2x_2 - (a_1 + a_2)x_3; 2\Big) = \begin{cases} 4(a_1 + a_2) - 1 & \text{if } a_1 = 1,\ 4 \mid a_2 \text{ or } (a_1, a_2) = (3, 4); \\ 4(a_1 + a_2) + 1 & \text{otherwise.} \end{cases}$$

2 Preliminaries

Let $\mathbb{N}$ be the set of positive integers and let $\chi : [a, b] \to [0, t-1]$ denote a t-colouring of numbers in the interval $\{n \in \mathbb{N} : a \leq n \leq b\}$. Given a t-colouring χ and a system of linear equations in m variables, a solution $(x_1, x_2, .., x_m)$ to the system is said to be monochromatic if and only if $\chi(x_i) = \chi(x_j)$ for every i and j pair. We call a monochromatic solution *non-trivial* if all x_i are distinct.

Definition 1. *For $t \geq 1$, a linear equation $\mathcal{L}$ is called $t-regular$ if there exists $n = \mathcal{R}(\mathcal{L}, t)$ such that for every $t-colouring$ of $[1, n]$ there is a monochromatic solution to $\mathcal{L}$. It is called regular if it is t-regular for all $t \geq 1$.*

Theorem 1. *(Rado's Single Equation Theorem) Let $\mathcal{L}$ represent the linear equation $\sum_{i=1}^{n} a_ix_i = 0$, where $a_i \in \mathbb{N}$ for $1 \leq i \leq n$. Then $\mathcal{L}$ is regular if and only if some nonempty subset of the a_i's sums to 0.*

The smallest number $\mathcal{R}(\mathcal{L}; t)$ such that any t-colouring of $[1, \mathcal{R}(\mathcal{L}; t)]$ admits a monochromatic solution to the equation $\mathcal{L}$: $a_1x_1 + \ldots + a_mx_m = 0$ is called its rado number. Throughout this section, we consider the regular equation $a_1x_1 + a_2x_2 - (a_1 + a_2)x_3 = 0$. For notational convenience, we use a, b for a_1, a_2, and $n(a, b)$ for $\mathcal{R}\big(a_1x_1 + a_2x_2 - (a_1 + a_2)x_3; 2\big)$. We may assume that $\gcd(a, b) = 1$ without loss of generality as otherwise the equation can be simplified. We further take a to be odd in what follows, without loss of generality. If x, y, z are integers such that $ax + by = (a + b)z$, then

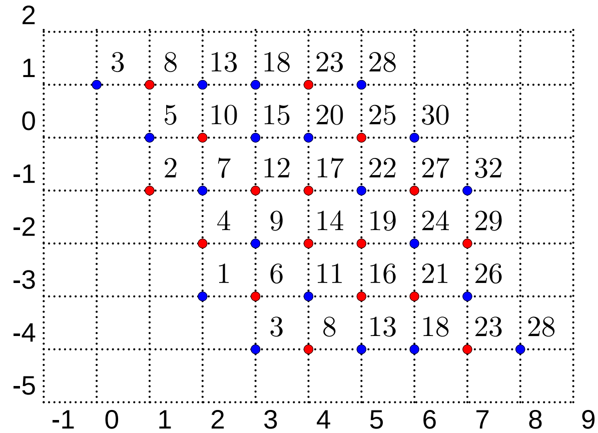


Figure 1: A snapshot of the infinite 2-dimensional grid with grid points (p,q) labeled with N(p,q) = 5p+3q for (a,b) = (5,3).

$y \equiv z \mod a$ and $x \equiv z \mod b$ since $\gcd(a, b) = 1$. Thus $y = z + ar$, $x = z + bs$ for some integers r, s; substituting in $ax + by = (a + b)z$, we have $r + s = 0$. Since $y = z + ar$, $x = z - br$ satisfies the equation $ax + by = (a + b)z$ for any $r \in \mathbb{Z}$, the given equation has the parametric solution

$$(x, y, z) \in \{(z - br, z + ar, z) : r \in \mathbb{Z}\}. \tag{1}$$

In an unpublished work, Burr & Loo [4] provided the upper bound $4(a + b) + 1$ for $n(a, b)$ for all (a,b) by showing that any colouring on the numbers $[1, 4(a + b) + 1]$ always yields a monochromatic solution. Landman & Robertson provide a different proof of this in [Theorem 9.12, [16]], and also show this to be sharp when $a = 1$ and $4 \nmid b$. To obtain the exact general bounds, we thus consider numbers in the interval $[1, 4(a + b)]$. For any solution $(x, y, z) \in [1, 4(a + b)]$, $\max\{x, y, z\} - \min\{x, y, z\} = |x - y| = (a + b)r$ implies $|r| \leq 3$ using (1). Note that $r \neq 0$ since x, y, z are distinct.

Burr & Loo introduced a rather interesting way to depict the colouring over [1,4(a+b)+1] and identify the solutions to the equation $ax + by = (a + b)z$ by relating the parametric form (1) to points in a 2-dimensional grid. We extend their representation to the infinite 2-dimensional grid and label every point (p, q) in $\mathbb{Z}^2$ with the number $N(p, q) = ap + bq$. Note that multiple points on the grid maybe mapped to the same number. For example, in Figure 1 points $(0, 1)$ and $(3, -4)$ both map to the number 3. Whenever a colouring on $[1, 4(a + b)]$ induces a monochromatic solution (x, y, z) to $ax + by = (a + b)z$, there exist monochromatic isosceles triangles right-angled at each point in $\{(p, q) : N(p, q) = z\}$ in $\mathbb{Z}^2$. A colouring $\lambda : \mathbb{Z}^2 \to \{0, 1\}$ maps naturally to a valid colouring $\chi : \mathbb{Z} \to \{0, 1\}$ if $\lambda((p_1, q)) = \lambda((p_2, q_2))$ whenever $N(p_1, q_1) = N(p_2, q_2)$.

We use the notation $AP(a, d; k)$ to denote the k-term arithmetic progression with first term a and common difference d.

3 Main Results

We begin by showing that any 2-colouring of the integers in $[1, M]$, where $M = \max\{2a+4b, 4a+2b\}$, in which three consecutive terms of an arithmetic progression with common difference a, b or $a + b$ are monochromatic, automatically gives a non-trivial monochromatic solution to $ax + by = (a + b)z$.

Lemma 1. *Let a, b be relatively prime positive integers, and let $M = \max\{2a+4b, 4a+2b\}$. For any $N \geq M$ and any $\chi : [1, N] \to \{0, 1\}$ such that $\chi(t) = \chi(t+d) = \chi(t+2d)$ for some $d \in \{a, b, a+b\}$, χ induces a non-trivial monochromatic solution to $ax + by = (a+b)z$.*

Proof. Let $\chi : [1, N] \to \{0, 1\}$, where $N \geq M = \max\{2a+4b, 4a+2b\}$. We consider the three cases in order and make repeated use of (1).
Suppose $\chi(t) = \chi(t+a) = \chi(t+2a) = \epsilon \in \{0, 1\}$. Suppose first that $t \geq 2b+1$. If either $\chi(t-b) = \epsilon$, $\chi(t+a-b) = \epsilon$ or $\chi(t-2b) = \epsilon$, then we get corresponding monochromatic solutions $\{t-b, t, t+a\}$, $\{t+a-b, t+a, t+2a\}$ and $\{t-2b, t, t+2a\}$. Hence, $\chi(t-b) = \chi(t+a-b) = \chi(t-2b) = 1-\epsilon$ which again gives a non-trivial monochromatic solution. Similarly, in the case $t \leq 2b$, if either $\chi(t+a+b) = \epsilon$, $\chi(t+2a+b) = \epsilon$ or $\chi(t+2a+2b) = \epsilon$, then we get corresponding monochromatic solutions $\{t, t+a, t+a+b\}$, $\{t+a, t+2a, t+2a+b\}$ and $\{t, t+2a, t+2a+2b\}$. Thus, forcing $\chi(t+a+b) = \chi(t+2a+b) = \chi(t+2a+2b) = 1-\epsilon$, which again gives a non-trivial monochromatic solution.
The same argument interchanging the roles of a and b prove the assertion for arithmetic progressions with common difference b.
Next, consider the case when $\chi(t) = \chi(t+a+b) = \chi(t+2a+2b) = \epsilon$ for $\epsilon \in \{0, 1\}$. If any one of $\chi(t+a), \chi(t+2a), \chi(t+2a+b)$ equals ϵ, we have a monochromatic solution, since the triples $\{t, t+a, t+a+b\}$, $\{t, t+2a, t+2a+2b\}$, and $\{t+a+b, t+2a+b, t+2a+2b\}$ are all solutions. Hence $\chi(t+a) = \chi(t+2a) = \chi(t+2a+b) = 1-\epsilon$, and we have a monochromatic solution. ■

Lemma 2. *For positive integers a, b, N, with $\gcd(a, b) = 1$ and $\min\{a, b\} > N$, let $T = \{ax+by+1 : 0 \leq x, y \leq N\}$. Then $\mathtt{f} : T \to T$ given by*

$$\mathtt{f}(ax+by+1) = bx+ay+1$$

is a bijection on T. Suppose $\chi : T \to \{0, 1\}$, and let (x_0, y_0, z_0) be a monochromatic solution to $ax + by = (a+b)z$ under χ with distinct x, y, z. Then $\big(\mathtt{f}(x_0), \mathtt{f}(y_0), \mathtt{f}(z_0)\big)$ is a monochromatic solution to $ax + by = (a+b)z$ with distinct x, y, z under $\chi \circ \mathtt{f} : T \to \{0, 1\}$ given by

$$(\chi \circ f)(n) = \chi\big(\mathtt{f}(n)\big).$$

Proof. Suppose $ax+by+1 = ax'+by'+1$, where $x, x', y, y' \in [0, N]$. Then $a(x-x') = -b(y-y')$, and since $\gcd(a, b) = 1$, we have $x-x' = bt$, $y-y' = -at$ for some $t \in \mathbb{Z}$. Since $\max\{|x-x'|, |y-y'|\} \leq N < \min\{a, b\}$, this implies $x = x'$ and $y = y'$. Thus there is a one-to-one correspondence between the set of lattice points in $[0, N] \times [0, N]$ and the set of integers in T, given by $(x, y) \mapsto ax+by+1$, and so $\mathtt{f}$ is well-defined. If $bx+ay+1 = bx'+ay'+1$, the same argument gives $x = x'$ and $y = y'$. Thus $\mathtt{f}$ is injective, and hence bijective since $\mathtt{f} : T \to T$. In fact, via the correspondence $T \leftrightarrow [0, N] \times [0, N]$, $\mathtt{f}$ maps (x, y) to (y, x), and is analogous to reflecting the lattice along the $x = y$ line.

Let (x_0, y_0, z_0) be a monochromatic solution to $ax+by = (a+b)z$ under χ, with x_0, y_0, z_0 distinct integers in T. If $z_0 \leftrightarrow (r, s)$, then by (1), $x_0 \leftrightarrow (r, s-t)$ and $y_0 \leftrightarrow (r+t, s)$ for $t \in \{\pm 1, \pm 2, \pm 3\}$. Thus $\mathtt{f}(x_0) \leftrightarrow (s-t, r)$, $\mathtt{f}(y_0) \leftrightarrow (s, r+t)$, and $\mathtt{f}(z_0) \leftrightarrow (s, r)$. By (1), $\big(\mathtt{f}(y_0), \mathtt{f}(x_0), \mathtt{f}(z_0)\big)$ is again a solution to $ax + by = (a+b)z$ with distinct x, y, z. ■

Theorem 2. (Burr & Loo)

Every 2-colouring of $[1, 4(a+b)+1]$ admits a monochromatic solution to $ax + by = (a+b)z$ with x, y, z distinct integers.

We next prove the exact rado numbers for the equation $ax+by=(a+b)z$. The proof is divided into two broad cases, when $4 \nmid ab$ and the other $4 \mid ab$. In each case, we either give a colouring that avoids a monochromatic solution on $[1, 4a+4b]$, which using Theorem 2 implies that $n(a,b)=4(a+b)+1$, or we give a colouring that avoids a monochromatic solution on $[1, 4(a+b)-2]$ and show that every colouring on $[1, 4(a+b)-1]$ induces a monochromatic solution.

4 The Case $4 \nmid ab$

We first consider the case when $4 \nmid ab$. Since we take a to be odd, this means that b is either an odd integer or twice an odd integer.

Theorem 3. *Let a, b be relatively prime positive integers such that $4 \nmid ab$. Then there exists a 2-colouring of $[1, 4(a+b)]$ which admits no monochromatic solution to $ax+by=(a+b)z$ with x, y, z distinct integers. In particular, $n(a,b)=4(a+b)+1$.*

Proof. Since a is odd, we consider the two cases: (i) b is odd; (ii) $b \equiv 2 \mod 4$.

Case I: (b is odd) Define a colouring $\chi : [1, 4(a+b)] \to \{0,1\}$ as follows:

$$\chi(n)=\begin{cases} 0 & \text{if } n \in [1, 2(a+b)],\ n \text{ is even, or } n \in [2(a+b)+1, 4(a+b)],\ n \text{ is odd;} \\ 1 & \text{if } n \in [1, 2(a+b)],\ n \text{ is odd, or } n \in [2(a+b)+1, 4(a+b)],\ n \text{ is even.} \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by=(a+b)z$ with x, y, z distinct integers. Suppose $\chi(x)=\chi(y)=\chi(z)$, where x, y, z are distinct integers satisfying $ax+by=(a+b)z$. Since a, b are both odd, x, y have the same parity, and so both belong to $[1, 2(a+b)]$ or to $[2(a+b)+1, 4(a+b)]$. But then z must lie in the same interval as x, y (since it lies between x and y), and so must have the same parity (since it has the same colour), implying r is even. But then $r=\pm 2$, and this is a contradiction since $|x-y|=2(a+b)$ is greater than either interval length.

An alternate proof for this theorem can be given using the grid framework. We refer to this colouring as the "diagonal" colouring. Consider the infinite grid restricted to numbers from 1 to $4(a+b)$. The diagonal colouring colours each minor diagonal (with at most 4 grid points), alternately with 0011 and 1100, i.e if $\chi(x, x+a+b, x+2(a+b), x+3(a+b))=(0,0,1,1)$ then $\chi(x+b, x+a+2b, x+2a+3b, x+3a+4b)=(1,1,0,0)$ and so on. Proving that such a colouring works entails showing that (1) there cannot exist monochromatic triangles that correspond to a solution, and (2) every point (p,q) that maps to the same number is coloured with the same colour. The grid shown in Figure 1 depicts a grid for $(a,b)=(5,3)$ that is coloured using the *diagonal* colouring.

Case II: $\big(b \equiv 2 \mod 4\big)$ Define a colouring $\chi : [1, 4(a+b)] \to \{0,1\}$ as follows:

$$\chi(n)=\begin{cases} 0 & \text{if } n \equiv 1 \text{ or } 2 \mod 4; \\ 1 & \text{if } n \equiv 0 \text{ or } 3 \mod 4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by=(a+b)z$ with x, y, z distinct integers. Suppose $\chi(x)=\chi(y)=\chi(z)$, where x, y, z are distinct integers satisfying $ax+by=(a+b)z$. Since x, z have the same parity and the same colour, $4 \mid (x-z)=br$. Thus r is even, hence $r=\pm 2$, and y, z also have the same parity. But then $4 \mid (y-z)$, and this is false since $|y-z|=2a$. ■

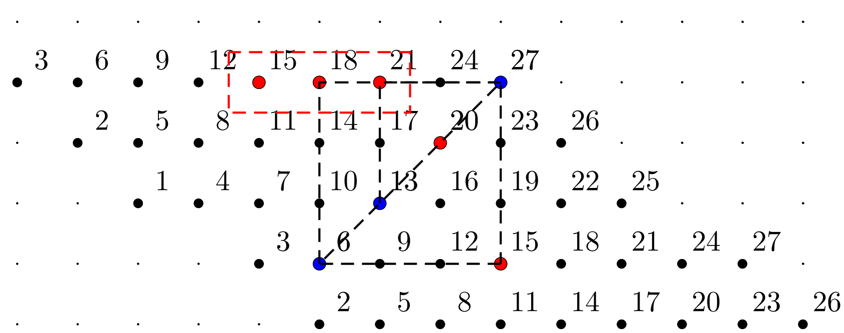
5 The Case $4 \mid ab$

5.1 The subcase $a = 3, b = 4$

Theorem 4. *Every 2-colouring of* $[1, 27]$ *admits a monochromatic solution to* $3x + 4y = 7z$ *with* x, y, z *distinct integers.*

Proof. Suppose that $\chi : [1, 27] \to \{0, 1\}$ is a colouring which admits no monochromatic solution to $3x + 4y = 7z$ with x, y, z distinct integers. Consider the colouring on the elements of $S = AP(6, 7; 4) = \{6, 13, 20, 27\}$. By Lemma 1, the only cases to consider for the ordered 4-tuple $\big(\chi(6), \chi(13), \chi(20), \chi(27)\big)$ are (i) $(0, 0, 1, 0)$, (ii) $(0, 0, 1, 1)$, (iii) $(0, 1, 0, 0)$, (iv) $(0, 1, 0, 1)$ and (v) $(0, 1, 1, 0)$. We repeatedly use (1) and Lemma 1 to prove the five cases.

- *Case* (i): Note that $\chi(6) = \chi(27) = 0$ forces $\chi(15) = \chi(18) = 1$, and that $\chi(13) = \chi(27) = 0$ forces $\chi(21) = 1$. Now $\{15, 18, 21\}$ forces a monochromatic solution by Lemma 1. Each part in this proof can be easily checked using the grid representation. We depict only the first part.



- *Case* (ii): Note that $\chi(6) = \chi(13) = 0$ forces $\chi(9) = \chi(10) = 1$, and that $\chi(20) = \chi(27) = 1$ forces $\chi(23) = \chi(24) = 0$.

  We claim that $\chi(17) = 0$. Otherwise, $\chi(10) = \chi(17) = 1$ forces $\chi(3) = 0$, which together with $\chi(24) = 0$ forces $\chi(15) = 1$. Now $\chi(9) = \chi(15) = 1$ forces $\chi(12) = 0$, leading to the monochromatic solution $\{3, 12, 24\}$. Thus $\chi(17) = 0$. Together with $\chi(13) = 0$, this forces $\chi(21) = 1$.

  If $\chi(15) = 0$, then $\chi(24) = 0$ forces $\chi(3) = 1$, which together with $\chi(10) = 1$ forces $\chi(7) = 0$. Now $\chi(7) = \chi(15) = 0$ forces $\chi(1) = 1$, which together with $\chi(10) = 1$ forces $\chi(22) = 0$. Next $\chi(15) = \chi(22) = 0$ forces $\chi(8) = \chi(18) = 1$, and $\chi(18) = \chi(21) = 1$ forces $\chi(25) = 0$. Finally $\chi(13) = \chi(25) = 0$ forces $\chi(4) = 1$, leading to the monochromatic solution $\{1, 4, 8\}$.

  If $\chi(15) = 1$, then $\chi(21) = 1$ forces $\chi(18) = 0$ and $\chi(9) = 1$ forces $\chi(1) = \chi(12) = 0$. Now $\chi(12) = \chi(18) = 0$ forces $\chi(4) = \chi(26) = 1$. Next $\chi(1) = \chi(13) = 0$ forces $\chi(22) = 1$, which together with $\chi(26) = 1$ forces $\chi(19) = 0$. Next $\chi(12) = \chi(19) = 0$ forces $\chi(5) = 1$, which together with $\chi(26) = 1$ forces $\chi(14) = 0$ and together with $\chi(9) = 1$ forces $\chi(2) = 0$. This gives the monochromatic solution $\{2, 14, 23\}$.

- *Case* (iii): Note that $\chi(6) = \chi(27) = 0$ forces $\chi(15) = \chi(18) = 1$, and that $\chi(6) = \chi(20) = 0$ forces $\chi(12) = 1$. Now $\{12, 15, 18\}$ forces a monochromatic solution by Lemma 1.

- *Case* (iv): Note that $\chi(6) = \chi(20) = 0$ forces $\chi(12) = \chi(14) = 1$, and that $\chi(13) = \chi(27) = 1$ forces $\chi(19) = \chi(21) = 0$.

If $\chi(11) = 0$, then $\chi(19) = 0$ forces $\chi(15) = \chi(25) = 1$. Now $\chi(12) = \chi(15) = 1$ forces $\chi(18) = 0$, which together with $\chi(11) = 0$ forces $\chi(4) = 1$. This leads to the monochromatic solution $\{4, 13, 25\}$.

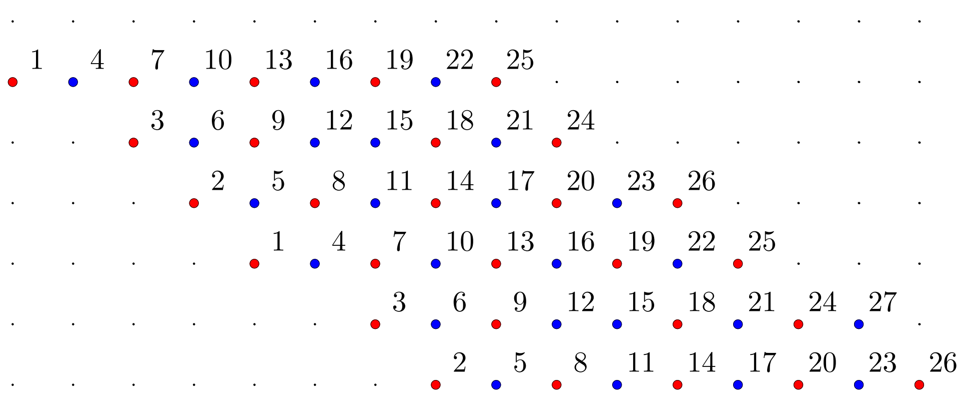
If $\chi(11) = 1$, then $\chi(14) = 1$ forces $\chi(7) = \chi(17) = 0$. Now $\chi(7) = \chi(21) = 0$ forces $\chi(15) = 1$, which together with $\chi(12) = 1$ forces $\chi(9) = 0$. Also $\chi(17) = \chi(21) = 0$ forces $\chi(24) = 1$, which together with $\chi(12) = 1$ forces $\chi(3) = 0$. This leads to the monochromatic solution $\{3, 9, 17\}$.

- *Case* (v): Note that $\chi(6) = \chi(27) = 0$ forces $\chi(15) = \chi(18) = 1$, and that $\chi(13) = \chi(20) = 1$ forces $\chi(16) = \chi(17) = 0$. Now $\chi(15) = \chi(18) = 1$ forces $\chi(21) = 0$, which together with $\chi(17) = 0$ gives $\chi(14) = \chi(24) = 1$. Next $\chi(14) = \chi(18) = 1$ forces $\chi(10) = 0$, which together with $\chi(17) = 0$ forces $\chi(3) = 1$. This leads to the monochromatic solution $\{3, 15, 24\}$.

■

Theorem 5. *There exists a 2-colouring of* $[1, 26]$ *which admits no monochromatic solution to* $3x + 4y = 7z$ *with* x, y, z *distinct integers. In particular,* $n(3, 4) = 27$.

Proof. We give the colouring on the grid for $(a, b) = (3, 4)$. As a quick check, the reader can observe



that this colouring does not admit any monochromatic solution triangles. More generally, we provide a 2-colouring of $[1, 4(a + b) - 2]$ which admits no monochromatic solution to $ax + by = (a + b)z$ with x, y, z distinct integers for the case $a = b-1$, $4 \mid b$. Note that $4(a+b)-2 = 8b-6$, and set $N = 8b-5$. Define a colouring $\chi : [1, 8b - 6] \to \{0, 1\}$ by first defining it on the interval $[1, 4b - 3]$ by

$$\chi(n) = \begin{cases} 0 & \text{if } \lceil \frac{n}{b-1} \rceil \text{ is even;} \\ 1 & \text{if } \lceil \frac{n}{b-1} \rceil \text{ is odd,} \end{cases}$$

and then extending it to the interval $[1, 8b-6]$ by $\chi(n) = \chi(N-n)$ for $4b-3 \lneq n \le 8b-6$. Observe that the mapping $x \mapsto (4b-3)-x$ sends the interval $I_t = [(t-1)(b-1)+1, t(b-1)]$ to the interval $J_t = [(4-t)(b-1)+1, (5-t)(b-1)]$ for $1 \le t \le 4$, so that $\lceil \frac{i}{b-1} \rceil + \lceil \frac{4b-3-i}{b-1} \rceil = 5$ for each $i \in I_t$, $1 \le t \le 4$. We show that χ admits no monochromatic solution to $(b-1)x + by = (2b-1)z$ with x, y, z distinct integers.

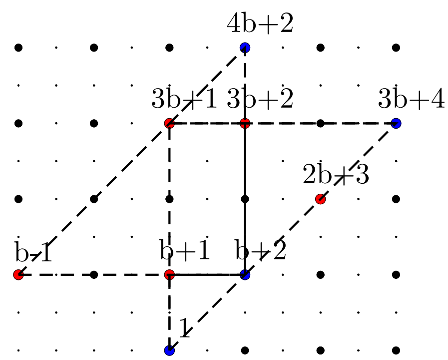
Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $(b-1)x + by = (2b-1)z$. Since y, z have the same colouring, from (1), $|y - z| = (b-1)|r|$ and so $r = \pm 2$. Thus $|x - y| = 2(2b-1) = N - (4b-3)$; without loss of generality, suppose $x < y$. Now $\chi(x) = \chi(y) = \chi(N-y) = \chi(4b-3-x)$, so that if $x \in I$, then $4b-3-x \in J$. This leads to a contradiction since $\lceil \frac{x}{b-1} \rceil + \lceil \frac{4b-3-x}{b-1} \rceil = 5$ is odd. ■

5.2 The subcase $a = 1$

Theorem 6. *Suppose $a = 1$ and $4 \mid b$. Then every 2-colouring of $[1, 4b+3]$ admits a monochromatic solution to $x + by = (b+1)z$ with x, y, z distinct integers.*

Proof. Suppose that $\chi : [1, 4b+3] \to \{0, 1\}$ is a colouring which admits no monochromatic solution to $x + by = (b+1)z$ with x, y, z distinct integers. Consider the colouring on the elements of $S = AP(1, b+1; 4)$. By Lemma 1, the only cases to consider for the ordered 4-tuple $\big(\chi(1), \chi(b+2), \chi(2b+3), \chi(3b+4)\big)$ are (i) $(0,0,1,0)$, (ii) $(0,0,1,1)$, (iii) $(0,1,0,0)$, (iv) $(0,1,0,1)$ and (v) $(0,1,1,0)$. We repeatedly use Lemma 1 to prove the five cases.

- *Case* (i). Note that $\chi(1) = \chi(b+2) = 0$ forces $\chi(b+1) = 1$, that $\chi(b+2) = \chi(3b+4) = 0$ forces $\chi(3b+2) = 1$, and that $\chi(1) = \chi(3b+4) = 0$ forces $\chi(3b+1) = 1$. Now $\chi(3b+1) = \chi(3b+2) = 1$ forces $\chi(4b+2) = 0$, which together with $\chi(b+2) = 0$ forces $\chi(b-1) = 1$. But now $\{b-1, b+1, 3b+1\}$ gives a monochromatic solution. We again show this part on the grid, the remaining parts can be checked similarly.



- *Case* (ii). Note that $\chi(1) = \chi(b+2) = 0$ forces $\chi(2) = \chi(b+1) = 1$ and $\chi(2b+3) = \chi(3b+4) = 1$ forces $\chi(2b+4) = \chi(3b+3) = 0$.

  Assume first that $\chi(b+3) = 0$. We claim that

$$\begin{aligned} \big(\chi(i), \chi(b+i), \chi(2b+i), \chi(3b+i)\big) &= (\epsilon, 1-\epsilon, \epsilon, 1-\epsilon) \\ \big(\chi(i+1), \chi(b+i+1), \chi(2b+i+1), \chi(3b+i+1)\big) &= (\epsilon, 1-\epsilon, \epsilon, 1-\epsilon) \end{aligned} \tag{2}$$

  for some $i \in [2, b]$ implies

$$\begin{aligned} \big(\chi(i+2), \chi(b+i+2), \chi(2b+i+2), \chi(3b+i+2)\big) &= (1-\epsilon, \epsilon, 1-\epsilon, \epsilon) \\ \big(\chi(i+3), \chi(b+i+3), \chi(2b+i+3), \chi(3b+i+3)\big) &= (1-\epsilon, \epsilon, 1-\epsilon, \epsilon). \end{aligned}$$

  Note that $\chi(b+3) = 0$ together with $\chi(b+2) = 0$ forces $\chi(b+4) = 1$, and with $\chi(2b+4) = 0$ forces $\chi(3b+5) = 1$, which together with $\chi(2) = 1$ forces $\chi(3b+2) = 0$. Next $\chi(b+2) = \chi(3b+2) = 0$ forces $\chi(2b+2) = 1$, and $\chi(3b+4) = \chi(3b+5) = 1$ forces $\chi(3b+6) = 0$, which together with $\chi(3b+3) = 0$ forces $\chi(3) = 1$. This proves (2) for $i = 2$.

  Assume (2) holds for some $i \in [2, b]$. Then $\chi(i) = \chi(i+1) = \epsilon$ forces $\chi(i+2) = 1-\epsilon$, and $\chi(b+i) = \chi(b+i+1) = 1-\epsilon$ forces $\chi(b+i+2) = \epsilon$. Next $\chi(2b+i) = \chi(2b+i+1) = \epsilon$ forces $\chi(2b+i+2) = 1-\epsilon$, and $\chi(3b+i) = \chi(3b+i+1) = 1-\epsilon$ forces $\chi(3b+i+2) = \epsilon$. Again $\chi(b+i+1) = \chi(3b+i+1) = 1-\epsilon$ forces $\chi(3b+i+3) = \epsilon$, which together with $\chi(i) = \epsilon$ forces

$\chi(i+3) = 1-\epsilon$. Now $\chi(i+2) = \chi(i+3) = 1-\epsilon$ forces $\chi(b+i+3) = \epsilon$, which together with $\chi(3b+i+3) = \epsilon$ forces $\chi(2b+i+3) = 1-\epsilon$.

Now assume that $\chi(b+3) = 1$. We claim that

$$\begin{aligned}\big(\chi(i), \chi(b+i+1), \chi(2b+i+2), \chi(3b+i+3)\big) &= (\epsilon, \epsilon, 1-\epsilon, 1-\epsilon), \\ \big(\chi(i+1), \chi(b+i+2), \chi(2b+i+3), \chi(3b+i+4)\big) &= (1-\epsilon, 1-\epsilon, \epsilon, \epsilon)\end{aligned} \tag{3}$$

for some $i \in [2, b-2]$ implies

$$\big(\chi(i+2), \chi(b+i+3), \chi(2b+i+4), \chi(3b+i+5)\big) = (\epsilon, \epsilon, 1-\epsilon, 1-\epsilon).$$

Note that $\chi(b+3) = 1$ together with $\chi(2b+3) = 1$ forces $\chi(3) = 0$, which together with $\chi(3b+3) = 0$ forces $\chi(3b+6) = 1$. Next $\chi(3b+4) = \chi(3b+6) = 1$ forces $\chi(3b+5) = 0$. This proves (3) for $i = 2$.

Assume (3) holds for some $i \in [1, b-1]$. Then $\chi(i) = \chi(b+i+1) = \epsilon$ forces $\chi(i+1) = 1-\epsilon$, and $\chi(2b+i+2) = \chi(3b+i+3) = 1-\epsilon$ forces $\chi(3b+i+2) = \epsilon$. Next $\chi(i+1) = \chi(b+i+2) = 1-\epsilon$ forces $\chi(i+2) = \epsilon$, and $\chi(2b+i+3) = \chi(3b+i+4) = \epsilon$ forces $\chi(2b+i+4) = 1-\epsilon$. Again $\chi(i+2) = \chi(3b+i+2) = \epsilon$ forces $\chi(3b+i+5) = 1-\epsilon$, which together with $\chi(2b+i+4) = 1-\epsilon$ forces $\chi(b+i+3) = \epsilon$.

Finally, we show that both (2) and (3) lead to a contradiction when $4 \mid b$. Since (2) implies $\chi(i+4) = \chi(i)$ for $i \geq b$, we must have $\chi(2b) = \chi(b)$ if $4 \mid b$, contradicting (2). Also since (3) holds for $i = 2$, we get $\chi(b, 2b+1, 3b+2, 4b+3) = (1, 1, 0, 0)$. If $\chi(b+1) = 0$, then we get the monochromatic solution $\{1, b+1, b+2\}$; if $\chi(b+1) = 1$, then we get the monochromatic solution $\{b, b+1, 2b+1\}$.

- *Case* (iii). Note that $\chi(1) = \chi(2b+3) = 0$ forces $\chi(3) = 1$, and that $\chi(1) = \chi(3b+4) = 0$ forces $\chi(4) = \chi(3b+1) = 1$. Now $\chi(3) = \chi(4) = 1$ forces $\chi(2) = 0$. Next $\chi(2b+3) = \chi(3b+4) = 0$ forces $\chi(2b+4) = \chi(3b+3) = 1$, which together with $\chi(3b+1) = 1$ forces $\chi(3b+2) = 0$. But $\chi(2) = \chi(3b+2) = 0$ forces $\chi(3b+5) = 1$, which together with $\chi(2b+4) = 1$ forces $\chi(2b+5) = 0$. Now $\chi(3) = \chi(3b+3) = 1$ forces $\chi(3b+6) = 0$, which together with $\chi(2b+5) = 0$ forces $\chi(2b+6) = 1$. This gives the monochromatic solution $\{4, 2b+4, 2b+6\}$.

- *Case* (iv). Note that $\chi(1) = \chi(2b+3) = 0$ forces $\chi(3) = \chi(2b+1) = 1$ and $\chi(b+2) = \chi(3b+4) = 1$ forces $\chi(b+4) = \chi(3b+2) = 0$.

  We claim that

$$\begin{aligned}\big(\chi(i), \chi(b+i), \chi(2b+i), \chi(3b+i)\big) &= (\epsilon, 1-\epsilon, 1-\epsilon, \epsilon) \\ \big(\chi(i+1), \chi(b+i+1), \chi(2b+i+1), \chi(3b+i+1)\big) &= (1-\epsilon, 1-\epsilon, \epsilon, \epsilon)\end{aligned} \tag{4}$$

  for some $i \in [2, b]$ implies

$$\begin{aligned}\big(\chi(i+2), \chi(b+i+2), \chi(2b+i+2), \chi(3b+i+2)\big) &= (1-\epsilon, \epsilon, \epsilon, 1-\epsilon) \\ \big(\chi(i+3), \chi(b+i+3), \chi(2b+i+3), \chi(3b+i+3)\big) &= (\epsilon, \epsilon, 1-\epsilon, 1-\epsilon).\end{aligned}$$

  Assume first that $\chi(2) = 0$. Now $\chi(2) = \chi(3b+2) = 0$ forces $\chi(3b+5) = 1$, which together with $\chi(3b+4) = 1$ forces $\chi(3b+3) = 0$. Next $\chi(2b+3) = \chi(3b+3) = 0$ forces $\chi(b+3) = \chi(2b+2) = 1$. This proves (4) for $i = 2$.

  Now assume that $\chi(2) = 1$. Note that $\chi(2) = \chi(b+2) = 1$ forces $\chi(2b+2) = 0$, and $\chi(2) = \chi(3) = 1$ forces $\chi(b+3) = 0$. Next $\chi(2b+2) = \chi(2b+3) = 0$ forces $\chi(2b+4) = \chi(3b+3) = 1$. Finally $\chi(2) = \chi(2b+4) = 1$ forces $\chi(4) = 0$. This proves (4) for $i = 3$.

Assume (4) holds for some $i \in [2, b]$. Then $\chi(i) = \chi(3b+i) = \epsilon$ forces $\chi(3b+i+3) = 1-\epsilon$, and $\chi(i+1) = \chi(b+i+1) = 1-\epsilon$ forces $\chi(b+i+2) = \epsilon$. Next $\chi(2b+i+1) = \chi(3b+i+1) = \epsilon$ forces $\chi(3b+i+2) = 1-\epsilon$, which together with $\chi(3b+i+3) = 1-\epsilon$ forces $\chi(2b+i+2) = \epsilon$. Again $\chi(b+i+2) = \chi(2b+i+2) = \epsilon$ forces $\chi(i+2) = \chi(2b+i+3) = 1-\epsilon$. Now $\chi(i+1) = \chi(i+2) = 1-\epsilon$ forces $\chi(i+3) = \epsilon$, and $\chi(2b+i+3) = \chi(3b+i+3) = 1-\epsilon$ forces $\chi(b+i+3) = \epsilon$.

Since (4) implies $\chi(i+4) = \chi(i)$ and $\chi(i+2b) = 1-\chi(i)$, we have a contradiction when $4 \mid b$ except when $b = 4$ in the case when $\chi(2) = 1$. But in this case, $\chi(3) = \chi(6) = 1$ forces $\chi(18) = 0$ which gives a solution using Lemma 1, as $\chi(10) = \chi(14) = 0$.

- *Case* (v). Note that $\chi(1) = \chi(3b+4) = 0$ forces $\chi(4) = \chi(3b+1) = 1$ and $\chi(b+2) = \chi(2b+3) = 1$ forces $\chi(2b+2) = \chi(b+3) = 0$.

  Assume first that $\chi(3) = 0$. We claim that

$$\begin{aligned} \big(\chi(i), \chi(b+i), \chi(2b+i), \chi(3b+i)\big) &= (\epsilon, \epsilon, 1-\epsilon, 1-\epsilon) \\ \big(\chi(i+1), \chi(b+i+1), \chi(2b+i+1), \chi(3b+i+1)\big) &= (1-\epsilon, 1-\epsilon, \epsilon, \epsilon) \end{aligned} \tag{5}$$

  for some $i \in [2, b+1]$ implies

$$\big(\chi(i+2), \chi(b+i+2), \chi(2b+i+2), \chi(3b+i+2)\big) = (\epsilon, \epsilon, 1-\epsilon, 1-\epsilon).$$

  Note that $\chi(3) = \chi(b+3) = 0$ forces $\chi(2) = 1$, which together with $\chi(4) = 1$ forces $\chi(2b+4) = 0$. Next $\chi(b+3) = \chi(2b+4) = 0$ forces $\chi(3b+5) = 1$, which together with $\chi(2) = 1$ forces $\chi(3b+2) = 0$. Next $\chi(3b+2) = \chi(3b+4) = 0$ forces $\chi(3b+3) = 1$. This proves (5) for $i = 2$.

  Assume (5) holds for some $i \in [2, b+1]$. Then $\chi(b+i) = \chi(2b+i+1) = \epsilon$ forces $\chi(i-1) = \chi(3b+i+2) = 1-\epsilon$, which in turn force $\chi(i+2) = \epsilon$. Next $\chi(i+1) = \chi(b+i+1) = 1-\epsilon$ forces $\chi(b+i+2) = \epsilon$, and $\chi(i+2) = \chi(b+i+2) = \epsilon$ forces $\chi(2b+i+2) = 1-\epsilon$.

  Now assume that $\chi(3) = 1$. We claim that

$$\begin{aligned} \big(\chi(i), \chi(b+i), \chi(2b+i), \chi(3b+i)\big) &= (\epsilon, 1-\epsilon, \epsilon, 1-\epsilon), \\ \big(\chi(i+1), \chi(b+i+1), \chi(2b+i+1), \chi(3b+i+1)\big) &= (\epsilon, 1-\epsilon, \epsilon, 1-\epsilon) \end{aligned} \tag{6}$$

  for some $i \in [3, b]$ implies

$$\begin{aligned} \big(\chi(i+2), \chi(b+i+2), \chi(2b+i+2), \chi(3b+i+2)\big) &= (1-\epsilon, \epsilon, 1-\epsilon, \epsilon) \\ \big(\chi(i+3), \chi(b+i+3), \chi(2b+i+3), \chi(3b+i+3)\big) &= (1-\epsilon, \epsilon, 1-\epsilon, \epsilon). \end{aligned}$$

  Note that $\chi(3) = \chi(4) = 1$ forces $\chi(2) = \chi(5) = \chi(b+4) = 0$. Now $\chi(b+3) = \chi(b+4) = 0$ forces $\chi(2b+4) = 1$, and $\chi(3) = \chi(2b+3) = 1$ forces $\chi(2b+5) = 0$, which together with $\chi(b+4) = 0$ forces $\chi(3b+6) = 1$. Finally $\chi(3) = \chi(3b+6) = 1$ forces $\chi(3b+3) = 0$. This proves (6) for $i = 3$.

  Assume (6) holds for some $i \in [3, b]$. Using the argument in the first case of case (ii), we conclude that the claim given above holds.

  Finally, we show that both (5) and (6) lead to a contradiction when $4 \mid b$. Note that both (5) and (6) imply $\chi(i+4) = \chi(i)$ for $i \geq 3$. This contradicts (5) since it also implies $\chi(2b+2) = \chi(2)$, and contradicts (6) since this also implies $\chi(b+3) = \chi(3)$.

■

Theorem 7. *Suppose $a = 1$ and $4 \mid b$. There exists a 2-colouring of $[1, 4b + 2]$ which admits no monochromatic solution to $x + by = (b + 1)z$ with x, y, z distinct integers. In particular, $n(a, b) = 4b + 3$.*

Proof. We use the parametric solution given in (1). Let $I = [b + 1, 2b] \cup [2b + 3, 3b + 2]$. Define a colouring $\chi : [1, 4(b + 1) - 2] \to \{0, 1\}$ as follows:

$$\chi(n) = \begin{cases} 0 & \text{if } n \in I,\ n \text{ is even, or } n \notin I,\ n \text{ is odd;} \\ 1 & \text{if } n \in I,\ n \text{ is odd, or } n \notin I,\ n \text{ is even.} \end{cases}$$

We show that χ admits no monochromatic solution to $x+by = (b+1)z$ with x, y, z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $x + by = (b + 1)z$. If r is odd, then x, y have opposite parity, and so exactly one belongs to I. This implies $r = \pm 1$, since the maximum difference between elements in I and those not in I is $3b + 1$ whereas $|x - y| = (b + 1)|r|$. If $r = \pm 1$, then y, z are consecutive integers having the same colour, and so exactly one belongs to I. Thus z is one of $b, b + 1, 2b, 2b + 1, 2b + 2, 2b + 3, 3b + 2, 3b + 3$. Now since $|x - z| = b$ and both belong to I or both do not, the only possibility is $(x, z) = (3b, 2b)$. But then $y = 2b - 1$, and this is impossible since $x, y, z \in I$ in that case. If r is even, then $r = \pm 2$ and x, y, z have the same parity. Thus all or none of x, y, z belong to I, since they have the same colour. This is not possible if $x, y \in I$ since $(3b + 2) - (b + 1) < |x - y| = 2b + 2$. If $x, y \notin I$, then $\min\{x, y\} \in [1, b]$ since $|x-y| = 2b+2$. But then $\max\{x, y\} \in [2b+3, 3b+2] \subset I$, which is a contradiction to $x, y \notin I$. ∎

5.3 The subcase $a \neq 1$, $(a, b) \neq (3, 4)$

Definition 2. *We define a sequence $\{s_i\}_{i\geq 0}$ by*

$$s_{i+1} = \begin{cases} s_i - b & \text{if } s_i > b; \\ s_i + a & \text{if } s_i \leq b, \end{cases}$$

with $s_0 := 1$.

Lemma 3. *For positive and coprime integers a, b, let $\{s_i\}_{i\geq 0}$ be as in Definition 1, and let $S_i := AP(s_i, a + b; 4)$ for $i \geq 0$. Then the following hold:*

(i) *$s_0, \dots, s_{a+b-1}$ are distinct integers in $[1, a + b]$.*
(ii) *$s_{i+a+b} = s_i$ for $i \geq 0$.*
(iii) *$\{S_0, \dots, S_{a+b-1}\}$ partitions $[1, 4(a + b)]$.*

Proof.

(i) Write $s_i = s_0 + i_1 a - i_2 b$, $s_j = s_0 + j_1 a - j_2 b$, and note that $i = i_1 + i_2$, $j = j_1 + j_2$ and $s_i, s_j \in [1, a + b]$. Now $s_i = s_j$ if and only if $(i_1 - j_1)a = (i_2 - j_2)b$. Since $\gcd(a, b) = 1$, this is only possible when $a \mid (i_2 - j_2)$ and $b \mid (i_1 - j_1)$; set $i_1 - j_1 = bt$ and $i_2 - j_2 = at$, $t \in \mathbb{Z}$. Thus $i - j = (i_1 + i_2) - (j_1 + j_2) = (a + b)t$, which is only possible when $t = 0$ since $i, j \in [0, a + b - 1]$. Thus $s_0, \dots, s_{a+b-1}$ are distinct integers in $[1, a + b]$, and so $\{s_0, \dots, s_{a+b-1}\} = \{1, \dots, a + b\}$.

(ii) To show that $s_{i+a+b} = s_i$ for $i \geq 0$, it is enough to show that $s_{a+b} = s_0$. Write $s_{a+b} = s_0 + ia - jb$, where $i + j = a + b$. By part (i), $s_{a+b} = s_k$ for some $k \in [0, a + b - 1]$. With $s_k = s_0 + i'a - j'b$, $i' + j' = k$, the argument in part (i) gives $a + b - k = (i + j) - (i' + j') = (a + b)t$. Since $0 \leq k \leq a + b - 1$, we must have $k = 0$.

(iii) Observe that each S_i is a 4-subset of $[1, 4(a+b)]$, because the largest integer among $S_0, \dots, S_{a+b-1}$ is $(a+b)+3(a+b) = 4(a+b)$. If, for $i \neq j$, $m \in S_i \cap S_j$ then m is congruent to both s_i and s_j mod $(a+b)$. This is not possible, by part (i). Hence $S_i \cap S_j = \emptyset$ for $i \neq j$, and so $\left|\bigcup_{i=0}^{a+b-1} S_i\right| = \sum_{i=0}^{a+b-1} |S_i| = 4(a+b)$. Thus $\{S_0, \dots, S_{a+b-1}\}$ partitions $[1, 4(a+b)]$.

■

Remark 1. *We set $s_{i+a+b} = s_i$ and $S_{i+a+b} = S_i$ for each $i \in \mathbb{Z}$.*

Lemma 4. *Let a, b be positive integers, with $\gcd(a,b) = 1$. Let I be any interval of b consecutive integers. Then every integer in $[1, 4(a+b)]$ is uniquely expressible as $1+ai+bj$ with $i \in I$ and $j \in \mathbb{Z}$.*

Proof. For any $m \in [1, 4(a+b)]$, choose $i, j \in \mathbb{Z}$ such that $m - 1 = ai + bj$. By repeatedly applying the transformation $(i,j) \mapsto (i \pm b, j \mp a)$, we can uniquely choose $t \in \mathbb{Z}$ so that $i + bt \in I$. Thus every integer in $[1, 4(a+b)]$ is of the form $1 + ai + bj$ with $i \in I$ and $j \in \mathbb{Z}$. To prove uniqueness of representation, suppose $1 + ai + bj = 1 + ak + b\ell$ with $i, k \in I$. This implies $b \mid (i-k)$, since $\gcd(a,b) = 1$, and since $|i - k| < b$, we must have $i = k$ and $j = \ell$. ■

Remark 2. *The result of Lemma 4 also holds for any interval J of a consecutive integers, with $i \in \mathbb{Z}$ and $j \in J$.*

Let $ax_0 + by_0 = (a+b)z_0$, with $x_0, y_0, z_0 \in [1, 4(a+b)]$. By (1) and Lemma 3, $x_0, y_0 \in S_i$ for some $i \in \{0, \dots, a+b-1\}$. We subdivide the construction of a valid 2-colouring of $[1, 4(a+b)]$ into three subcases: (i) $\frac{b}{a} > 2$ or $\frac{a}{b} > 2$; (ii) $\frac{4}{3} < \frac{b}{a} < 2$ or $\frac{4}{3} < \frac{a}{b} < 2$; (iii) $1 < \frac{b}{a} < \frac{4}{3}$ or $1 < \frac{a}{b} < \frac{4}{3}$. For $0 \leq i \leq a+b-1$, we use the notation $\chi(S_i) = \chi\big(AP(s_i, a+b; 4)\big)$ to denote the ordered tuple $(\chi(s_i), \chi(s_i+a+b), \chi(s_i+2a+2b), \chi(s_i+3a+3b))$.

Lemma 5. *Let a, b be positive, coprime integers, and let $ax + by = (a+b)z$, with distinct x, y, z in $[1, 4(a+b)]$. Let i be such that $\left|S_i \cap \{x, y\}\right| = 2$. Then $z \in S_{i-r} \cup S_{i+r}$, where $r = \frac{|x-y|}{a+b} \leq 3$.*

Proof. Since $x, y \in S_i$, we can write $x = s_i + t_1(a+b)$, $y = s_i + t_2(a+b)$, where t_1, t_2 are distinct integers in $[0, 3]$. Let $|t_1 - t_2| = r$. By (1), $z = s_i + t_1 a + t_2 b$ or $s_i + t_2 a + t_1 b$. So $z \equiv s_i \pm br \mod a+b$. Now $s_{i+r} = s_i + ak - b(r-k)$ and $s_{i-r} = s_i - a\ell + b(r - \ell)$ for some $k, \ell \in [0, r]$. So $s_{i \pm r} \equiv s_i \mp br \mod a+b$, so that $z \equiv s_{i+r} \mod a+b$ or $z \equiv s_{i-r} \mod a+b$. Hence $z \in S_{i-r} \cup S_{i+r}$. ■

Lemma 6. *Let a, b be relatively prime positive integers. For $q \geq p$, consider a 2-colouring χ on $[1, 4(a+b)]$, such that for $n \in \bigcup_{i=p}^{q} S_i$,*

$$\chi(S_i) = \begin{cases} 0011 & \text{if } i \text{ is odd;} \\ 1100 & \text{if } i \text{ is even.} \end{cases}$$

Then χ admits no monochromatic solution in $\bigcup_{i=p}^{q} S_i$. Moreover, there is no monochromatic solution x_0, y_0, z_0 such that $\left|S_i \cap \{x_0, y_0\}\right| = 2$ and $i \in [p+1, q-1]$.

Proof. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax + by = (a+b)z$ and $x, y, z \in \bigcup_{i=p}^{q} S_i$. Define i such that $x, y \in S_i$, by (1). Thus $\{x, y\} = \{s_i, s_i + a + b\}$ or $\{s_i + 2a + 2b, s_i + 3a + 3b\}$. By (1), $z \in \{s_i + a, s_i + b\}$ in the first case, and $z \in \{s_i + 2a + 3b, s_i + 3a + 2b\}$ in the second case. Observe that $s_i + a$ is the first term in S_{i+1} if $s_i \leq b$ and the second term in S_{i+1} if $s_i > b$; in both cases, $\chi(s_i + a) \neq \chi(s_i)$. Again observe that $s_i + b$ is the first term in S_{i-1} if

$s_i \le a$ and the second term in S_{i-1} if $s_i > a$; in both cases, $\chi(s_i + b) \ne \chi(s_i)$. A similar argument applies to the cases $z = s_i + 2a + 3b$ and $z = s_i + 3a + 2b$. Thus $\chi(z) \ne \chi(x)$ in all cases, which is a contradiction.

The colouring defined by χ forces $r = 1$, by (1). Let i be such that $|S_i \cap \{x_0, y_0\}| = 2$. By Lemma 5, $i \notin [p+1, q-1]$. ■

Remark 3. *Note that χ in Lemma 6 reduces to the 2-colouring in the first case of Theorem 2.*

Lemma 7. *Let a, b be relatively prime positive integers, and let I be any interval consisting of b consecutive integers. For $q \ge p$, consider a 2-colouring χ on $[1, 4(a+b)]$, such that for $n = 1 + ai + bj \in \bigcup_{k=p}^{q} S_k$,*

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 0,3 \mod 4 \text{ or } i \text{ is even, } j \equiv 1,2 \mod 4; \\ 1 & \text{if } i \text{ is odd, } j \equiv 1,2 \mod 4 \text{ or } i \text{ is even, } j \equiv 0,3 \mod 4. \end{cases}$$

Then χ admits no monochromatic solution in $\bigcup_{i=p}^{q} S_i$. Moreover, there is no monochromatic solution x_0, y_0, z_0 such that $|S_i \cap \{x_0, y_0\}| = 2$ and $i \in [p+3, q-3]$.

Proof. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax + by = (a+b)z$ and $x, y, z \in \bigcup_{i=p}^{q} S_i$. By (1) and Lemma 4, we can write $z = 1 + ai + bj$, $x = 1 + ai + b(j-r)$, $y = 1 + a(i+r) + bj$, with $r \in [-3, 3]$. Now $\chi(y) = \chi(z)$ implies that r is even. But then this contradicts $\chi(x) = \chi(y)$.

Let i be such that $|S_i \cap \{x_0, y_0\}| = 2$. By Lemma 5, $i \notin [p+3, q-3]$. ■

Remark 4. *The result of Lemma 7 also holds for the colouring*

$$\chi_1(1+ai+bj) = \begin{cases} 0 & \text{if } i \equiv 0,3 \mod 4, \ j \text{ is odd or } i \equiv 1,2 \mod 4, \ j \text{ is even;} \\ 1 & \text{if } i \equiv 1,2 \mod 4, \ j \text{ is odd or } i \equiv 0,3 \mod 4, \ j \text{ is even.} \end{cases}$$

obtained by applying Lemma 2 to Lemma 7. The result of Lemma 7 also holds for the colourings

$$\chi_2(1+ai+bj) = \begin{cases} 0 & \text{if } i \equiv 0,1 \mod 4, \ j \text{ is even or } i \equiv 2,3 \mod 4, \ j \text{ is odd;} \\ 1 & \text{if } i \equiv 0,1 \mod 4, \ j \text{ is odd or } i \equiv 2,3 \mod 4, \ j \text{ is even.} \end{cases}$$

and

$$\chi_3(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 2,3 \mod 4 \text{ or } i \text{ is even, } j \equiv 0,1 \mod 4; \\ 1 & \text{if } i \text{ is odd, } j \equiv 0,1 \mod 4 \text{ or } i \text{ is even, } j \equiv 2,3 \mod 4. \end{cases}$$

Remark 5. *Note that χ_2 in Remark 4 reduces to the 2-colouring in the second case of Theorem 2.*

We are now in a position to state and prove the result of the remaining case. The proof uses explicitly the sequence given in Definition 2. We need to consider three cases: (i) $\frac{b}{a}$ or $\frac{a}{b} \in (1, \frac{4}{3})$, (ii) $\frac{b}{a}$ or $\frac{a}{b} \in (\frac{4}{3}, 2)$ and (iii) $\frac{b}{a}$ or $\frac{a}{b} > 2$. Cases (i) and (iii) are further subdivided into two subcases: (i) $a \equiv 1 \mod 4$ and (ii) $a \equiv 3 \mod 4$. Our proof involves a rather cumbersome case-by-case listing of colourings as we have been unable to combine these colourings in a more meaningful manner.

Theorem 8. *Let a, b be relatively prime positive integers such that $4 \mid b$. If $a \ne 1$ and $(a, b) \ne (3, 4)$, there exists a 2-colouring of $[1, 4(a+b)]$ which admits no monochromatic solution to $ax + by = (a+b)z$ with x, y, z distinct integers. In particular, $n(a, b) = 4(a+b) + 1$.*

Proof. Consider the sequence $\{s_0, \dots, s_{a+b}\}$. For each i, $0 \le i \le a+b$, let $S_i := AP(s_i, a+b; 4)$.

Case (i). ($1 < \frac{b}{a} < \frac{4}{3}$ or $1 < \frac{a}{b} < \frac{4}{3}$)

We first consider the case $a < b < \frac{4}{3}a$, and its two subcases: $a \equiv 1 \mod 4$, and $a \equiv 3 \mod 4$. Note that $b = \frac{4}{3}a$ is only possible when $b = 4$ and $a = 3$, which we have already excluded.

Subcase (i) If $a \equiv 1 \mod 4$, then $a \ge 5$ since we have already considered $a = 1$. The smallest possible value of $a+b$ that satisfies this case is 29, with $(a,b) = (13,16)$. Define $\chi : [1, 4(a+b)] \to \{0,1\}$ by first defining it on $\bigcup_{i=-6}^{0} S_i$ as follows.

S	$\chi(S)$
S_{-6}, S_{-2}, S_0	1100
S_{-5}, S_{-3}	0011
S_{-1}, S_{-4}	0110

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-6}^{0} S_i$, by Lemma 4, n can be uniquely expressed as $1 + ai + bj$ with $j \in [4, a+3]$. Define

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 0,1 \mod 4 \text{ or } i \text{ is even, } j \equiv 2,3 \mod 4; \\ 1 & \text{if } i \text{ is even, } j \equiv 0,1 \mod 4 \text{ or } i \text{ is odd, } j \equiv 2,3 \mod 4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x,y,z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x,y,z are distinct integers satisfying $ax+by = (a+b)z$. Define i such that $x, y \in S_i$. From Lemma 7, there is no monochromatic solution in $\bigcup_{i=1}^{a+b-4} S_i$. Note that since $\bigcup_{i=-10}^{-6} S_i$ and $\bigcup_{i=-1}^{6} S_i$ also satisfy the second colouring in this case for $j \in [2, a+1]$ and $j \in [5, a+4]$, respectively, there is no monochromatic solution in $\bigcup_{i=-10}^{-6} S_i$ and $\bigcup_{i=-1}^{6} S_i$. By (1) and Lemma 5, there remain the cases $i \in [-6, 1]$.

i	$\{x,y\}$	$\{\chi(x),\chi(y)\}$	z	χ(z)
-6	$\{1-3a+3b, 1+6b\}$	$\{1,0\}$	1+3b	0
-5	$\{1-2a+3b, 1+a+6b\}$	$\{0,1\}$	1+a+3b	0
-4	$\{1-2a+2b, 1+a+5b\}$	$\{0,0\}$	1-2a+5b,1+a+2b	1
-4	$\{1-a+3b, 1+4b\}$	$\{1,1\}$	1-a+4b,1+3b	0
-3	$\{1-a+2b, 1+3b\}$	$\{0,0\}$	1-a+3b,1+2b	1
-3	$\{1+a+4b, 1+2a+5b\}$	$\{1,1\}$	1+a+5b,1+2a+4b	0
-2	$\{1-a+b, 1+2b\}$	$\{1,1\}$	1-a+2b,1+b	0
-2	$\{1+a+3b, 1+2a+4b\}$	$\{0,0\}$	1+2a+3b,1+a+4b	1
-1	$\{1+b, 1+3a+4b\}$	$\{0,0\}$	1+4b,1+3a+b	1
-1	$\{1+a+2b, 1+2a+3b\}$	$\{1,1\}$	1+a+3b,1+2a+2b	0
0	$\{1, 1+3a+3b\}$	$\{1,0\}$	1+3b	0
1	$\{1+a, 1+4a+3b\}$	$\{0,1\}$	1+a+3b	0

In each case, the solution sets are not monochromatic.

Subcase (ii) If $a \equiv 3 \mod 4$, we further consider two cases: (a) $a < b < \frac{5}{4}a$, and (b) $\frac{5}{4}a < b < \frac{4}{3}a$. For $a < b < \frac{5}{4}a$, the smallest possible value of $a+b$ that satisfies this case is 15, with $(a,b) = (7,8)$. Define $\chi : [1, 4(a+b)] \to \{0,1\}$ by first defining it on $\bigcup_{i=-7}^{3} S_i$ as follows.

S	$\chi(S)$
$S_{-7}, S_{-3}, S_{-1}, S_2$	0011
S_{-6}, S_{-4}, S_0, S_3	1100
S_{-5}, S_{-2}, S_1	0110

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-7}^{3} S_i$, by Lemma 4, n can be uniquely expressed as $1 + ai + bj$ with $j \in [3, a+2]$. Define

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 0,3 \mod 4 \text{ or } i \text{ is even, } j \equiv 1,2 \mod 4; \\ 1 & \text{if } i \text{ is even, } j \equiv 0,3 \mod 4 \text{ or } i \text{ is odd, } j \equiv 1,2 \mod 4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by=(a+b)z$ with x, y, z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax+by=(a+b)z$. Define i such that $x, y \in S_i$. By Lemma 4, $n \in \bigcup_{i=-2}^{4} S_i$ can be uniquely expressed as $1 + ai + bj$ with $i \in [-1, b-2]$, and also satisfy the following colouring.

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \equiv 0,3 \mod 4,\ j \text{ is odd or } i \equiv 1,2 \mod 4,\ j \text{ is even}; \\ 1 & \text{if } i \equiv 0,3 \mod 4,\ j \text{ is even or } i \equiv 1,2 \mod 4,\ j \text{ is odd}. \end{cases}$$

From Lemma 7, there is no monochromatic solution in $\bigcup_{i=-2}^{4} S_i$ and in $\bigcup_{i=5}^{a+b-8} S_i$. By (1) and Lemma 5, there remain the cases $i \in [-10, 7] \setminus \{1\}$.

i	$\{x, y\}$	$\{\chi(x), \chi(y)\}$	z	χ(z)
-10	$\{1-5a+5b, 1-2a+8b\}$	$\{1, 0\}$	1-2a+5b	0
-9	$\{1-4a+5b, 1-a+8b\}$	$\{0, 1\}$	1-a+5b	0
-8	$\{1-4a+4b, 1-a+7b\}$	$\{0, 0\}$	1-a+4b	1
-8	$\{1-3a+5b, 1-2a+6b\}$	$\{1, 1\}$	1-2a+5b	0
-7	$\{1-2a+5b, 1-3a+4b\}$	$\{0, 0\}$	1-2a+4b,1-3a+5b	1
-7	$\{1-a+6b, 1+7b\}$	$\{1, 1\}$	1-a+7b,1+6b	0
-6	$\{1-3a+3b, 1-2a+4b\}$	$\{1, 1\}$	1-2a+3b,1-3a+4b	0
-6	$\{1-a+5b, 1+6b\}$	$\{0, 0\}$	1+5b,1-a+6b	1
-5	$\{1-2a+3b, 1+a+6b\}$	$\{0, 0\}$	1+a+3b,1-2a+6b	1
-5	$\{1-a+4b, 1+5b\}$	$\{1, 1\}$	1-a+5b,1+4b	0
-4	$\{1-2a+2b, 1-a+3b\}$	$\{1, 1\}$	1-a+2b,1-2a+3b	0
-4	$\{1+4b, 1+a+5b\}$	$\{0, 0\}$	1+a+4b,1+5b	1
-3	$\{1-a+2b, 1+3b\}$	$\{0, 0\}$	1-a+3b,1+2b	1
-3	$\{1+a+4b, 1+2a+5b\}$	$\{1, 1\}$	1+a+5b,1+2a+4b	0
-2	$\{1-a+b, 1+2a+4b\}$	$\{0, 0\}$	1+2a+b,1-a+4b	1
-2	$\{1+2b, 1+a+3b\}$	$\{1, 1\}$	1+a+2b,1+3b	0
-1	$\{1+b, 1+a+2b\}$	$\{0, 0\}$	1+a+b,1+2b	1
-1	$\{1+3a+4b, 1+2a+3b\}$	$\{1, 1\}$	1+3a+3b,1+2a+4b	0
0	$\{1, 1+3a+3b\}$	$\{1, 0\}$	1+3b	0
2	$\{1+2a, 1+5a+3b\}$	$\{0, 1\}$	1+5a	0
3	$\{1+2a-b, 1+3a\}$	$\{1, 1\}$	1+3a-b	0
3	$\{1+4a+b, 1+5a+2b\}$	$\{0, 0\}$	1+5a+b	1
4	$\{1+3a-b, 1+6a+2b\}$	$\{0, 0\}$	1+3a+2b	1
4	$\{1+4a, 1+5a+b\}$	$\{1, 1\}$	1+4a+b	0
5	$\{1+3a-2b, 1+6a+b\}$	$\{1, 0\}$	1+3a+b	0
6	$\{1+4a-2b, 1+7a+b\}$	$\{0, 1\}$	1+4a+b	0

Note that for $(a,b)=(7,8)$, $S_{-10}=S_5$ and $S_{-9}=S_6$, but $\chi(S_{-10})=\chi(S_5)$ and $\chi(S_{-9})=\chi(S_6)$. Hence the above cases hold. In each case, the solution sets are not monochromatic.

For the case $\frac{5}{4}a<b<\frac{4}{3}a$, we shift the colouring χ used for the case $a<b<\frac{5}{4}a$. For $\frac{5}{4}a<b<\frac{4}{3}a$, the smallest possible value of $a+b$ that satisfies this case is 43, with $(a,b)=(19,24)$. Define $\chi':[1,4(a+b)]\to\{0,1\}$ by first defining it on $\bigcup_{i=-5}^{13}S_i$ as follows.

$$\chi'(S_i)=\begin{cases}\chi(S_{i-2}) & \text{if } i\in[-5,0];\\ \chi(S_{i-7}) & \text{if } i\in[3,13];\\ \ \ 1001 & \text{if } i=1;\\ \ \ 0110 & \text{if } i=2.\end{cases}$$

For $n\in[1,4(a+b)]\setminus\bigcup_{i=-5}^{13}S_i$, by Lemma 4, n can be uniquely expressed as $1+ai+bj$ with $i\in[8,b+7]$

$$\chi'(1+ai+bj)=\begin{cases}0 & \text{if } i \text{ is odd, } j\equiv 0,1 \mod 4 \text{ or } i \text{ is even, } j\equiv 2,3 \mod 4;\\ 1 & \text{if } i \text{ is odd, } j\equiv 2,3 \mod 4 \text{ or } i \text{ is even, } j\equiv 0,1 \mod 4.\end{cases}$$

We show that χ' admits no monochromatic solution to $ax+by=(a+b)z$ with x,y,z distinct integers. Suppose $\chi'(x)=\chi'(y)=\chi'(z)$, where x,y,z are distinct integers satisfying $ax+by=(a+b)z$. Define i such that $x,y\in S_i$. By Lemma 4, $n\in\bigcup_{i=-3}^{5}S_i$ can be uniquely expressed as $1+ai+bj$ with $i\in[-1,b-2]$, and also satisfy the following colouring.

$$\chi(1+ai+bj)=\begin{cases}0 & \text{if } i \text{ is even, } j\equiv 0,1 \mod 4 \text{ or } i \text{ is odd, } j\equiv 2,3 \mod 4;\\ 1 & \text{if } i \text{ is odd, } j\equiv 0,1 \mod 4 \text{ or } i \text{ is even, } j\equiv 2,3 \mod 4.\end{cases}$$

Note that S_{12},S_{13},S_{14} also satisfy the second colouring in this case. From Lemma 7, there is no monochromatic solution in $\bigcup_{i=-3}^{5}S_i$ and $\bigcup_{i=11}^{a+b-6}S_i$. Note that $\chi'(S_i)=\chi(S_{i-2})$ for $i\in\{-8,-7,-6\}$. For $i\in[-8,0]$, solutions $\{x,y,z\}$ under χ' correspond to solutions $\{x-a+b,y-a+b,z-a+b\}$ under χ; for $i\in[3,15]$, solutions $\{x,y,z\}$ under χ' correspond to solutions $\{x-4a+3b,y-4a+3b,z-4a+3b\}$ under χ. Hence there can be no monochromatic solution when $x,y,z\in\bigcup_{i=-8}^{0}S_i$ and in $\bigcup_{i=3}^{15}S_i$. There remain the cases $i\in\{3,5\}$.

i	$\{x,y\}$	$\{\chi(x),\chi(y)\}$	z	χ(z)
3	$\{1+2a-b,1+3a\}$	$\{1,1\}$	1+2a	0
3	$\{1+4a+b,1+5a+2b\}$	$\{0,0\}$	1+4a+2b	1
5	$\{1+3a-2b,1+6a+b\}$	$\{0,0\}$	1+6a-2b	1
5	$\{1+4a-b,1+5a\}$	$\{1,1\}$	1+5a-b	0

In each case, the solution sets are not monochromatic.

We now consider the case $b<a<\frac{4}{3}b$. For $b<a<\frac{4}{3}b$, the smallest possible value of $a+b$ that satisfies this case is 9, with $(a,b)=(5,4)$. By Lemma 4, n can be uniquely expressed as $1+ai+bj$ with $i\in[-1,b-2]$. Define $\chi:[1,4(a+b)]\to\{0,1\}$ by first defining it on $\bigcup_{i=-4}^{2}S_i$ as follows. Define

$$\chi(1+ai+bj)=\begin{cases}0 & \text{if } i \text{ is even, } j\equiv 0,3 \mod 4 \text{ or } i \text{ is odd, } j\equiv 1,2 \mod 4;\\ 1 & \text{if } i \text{ is odd, } j\equiv 0,3 \mod 4 \text{ or } i \text{ is even, } j\equiv 1,2 \mod 4.\end{cases}$$

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-4}^{2} S_i$, by Lemma 4, n can be uniquely expressed as $1 + ai + bj$ with $i \in [3, b+2]$

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \equiv 1,2 \mod 4,\ j \text{ is even or } i \equiv 0,3 \mod 4,\ j \text{ is odd;} \\ 1 & \text{if } i \equiv 1,2 \mod 4,\ j \text{ is odd or } i \equiv 0,3 \mod 4,\ j \text{ is even.} \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x, y, z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax+by = (a+b)z$. Define i such that $x, y \in S_i$. From Lemma 7, there is no monochromatic solution in $\bigcup_{i=3}^{a+b-5} S_i$ and in $\bigcup_{i=-4}^{2} S_i$. By (1) and Lemma 5, there remain the cases $i \in [-7, 5] \setminus \{-1\}$. Since $\chi(S_i) \in \{0011, 1100\}$ for $i \in \{-6, -3, -2, 0, 1, 4\}$, $r \neq 3$ in these cases, and so there remain the cases $i \in \{-7, -5, -4, 2, 3, 5\}$.

i	$\{x, y\}$	$\{\chi(x), \chi(y)\}$	z	χ(z)
-7	$\{1-3a+4b, 1+7b\}$	$\{1,1\}$	1+4b	0
-5	$\{1-2a+3b, 1-a+4b\}$	$\{0,0\}$	1-a+3b	1
-5	$\{1+5b, 1+a+6b\}$	$\{1,1\}$	1+a+5b	0
-4	$\{1-a+3b, 1+2a+6b\}$	$\{1,1\}$	1-a+6b	0
-4	$\{1+4b, 1+a+5b\}$	$\{0,0\}$	1+5b	1
2	$\{1+a-b, 1+4a+2b\}$	$\{1,1\}$	1+4a-b	0
2	$\{1+2a, 1+3a+b\}$	$\{0,0\}$	1+3a	1
3	$\{1+2a-b, 1+3a\}$	$\{1,1\}$	1+2a	0
3	$\{1+4a+b, 1+5a+2b\}$	$\{0,0\}$	1+4a+2b	1
5	$\{1+3a-2b, 1+6a+b\}$	$\{1,1\}$	1+3a+b	0

Note that for $(a, b) = (5, 4)$, $S_{-7} = S_2$ and $S_{-5} = S_4$. But $\chi(S_{-7}) = \chi(S_2)$ and $\chi(S_{-5}) = \chi(S_4)$. Hence the above cases hold. In each case, the solution sets are not monochromatic.

Case (ii). ($\frac{4}{3} < \frac{b}{a} < 2$ or $\frac{4}{3} < \frac{a}{b} < 2$)

We consider the case $\frac{4}{3} < \frac{b}{a} < 2$. For $\frac{4}{3} < \frac{b}{a} < 2$, the smallest possible value of $a+b$ that satisfies this case is 13, with $(a, b) = (5, 8)$. The argument for the other case is obtained by interchanging the roles of a and b, and is omitted. Recall that $\{S_0, \ldots, S_{a+b-1}\}$ partitions $[1, 4(a+b)]$, by Lemma 3. Set $S_{f(j)} = AP(j, a+b; 4)$, with $j = 1$ if $b < \frac{3}{2}a$ and $j = b-a$ if $b > \frac{3}{2}a$. Define $\chi : [1, 4(a+b)] \to \{0, 1\}$ by

$$\chi(\S_i) = \begin{cases} 0011 & \text{if } i + f(j) \text{ is odd, } i \in [f(j)+5, a+b+f(j)+2]; \\ 1100 & \text{if } i + f(j) \text{ is even, } i \in [f(j)+5, a+b+f(j)+2]; \\ 0110 & \text{if } i = f(j)+3; \\ 1001 & \text{if } i = f(j)+4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x, y, z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax+by = (a+b)z$. Define i such that $x, y \in S_i$. From Lemmas 5 and 6, it follows that there can be no monochromatic solution if $i \in [f(j)+6, a+b+f(j)+1]$. Now suppose $i \in \{f(j)+2, f(j)+3, f(j)+4, f(j)+5\}$. Note that $s_2 = 2a+j$, $s_3 = 2a-b+j$, $s_4 = 3a-b+j$, and $s_5 = 3a-2b+j$ in this case.

i	{x,y}	$\{\chi(x),\chi(y)\}$	z	χ(z)
f(j)+2	{3a+b+j,2a+j}	{0,0}	3a+j	1
f(j)+2	{5a+3b+j,4a+2b+j}	{1,1}	5a+2b+j	0
f(j)+3	{2a-b+j,5a+2b+j}	{0,0}	2a+2b+j,5a-b+j	1
f(j)+3	{3a+j,4a+b+j}	{1,1}	3a+b+j,4a+j	0
f(j)+4	{6a+2b+j,3a-b+j}	{1,1}	6a-b+j,3a+2b+j	0
f(j)+4	{5a+b+j,4a+j}	{0,0}	5a+j,4a+b+j	1
f(j)+5	{4a-b+j,3a-2b+j}	{0,0}	4a-2b+j	1
f(j)+5	{5a+j,6a+b+j}	{1,1}	5a+b+j	0

In each case, the solution sets are not monochromatic.

Case (iii). ($\frac{b}{a} > 2$ or $\frac{a}{b} > 2$)

We first consider the case $b > 2a$, and its two subcases: $a \equiv 1 \mod 4$, and $a \equiv 3 \mod 4$. Set $M = \lfloor \frac{2b}{a} \rfloor$.

Subcase (i) If $a \equiv 1 \mod 4$, then the smallest possible value of $a+b$ that satisfies this case is 17, with $(a,b) = (5,12)$. Define $\chi : [1, 4(a+b)] \to \{0,1\}$ by

$$\chi(S_i) = \begin{cases} 0011 & \text{if } i \in [-3, M+3] \text{ and } i \text{ is odd;} \\ 1100 & \text{if } i \in [-3, M+3] \text{ and } i \text{ is even.} \end{cases}$$

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-3}^{M+3} S_i$, by Lemma 4, n can be uniquely expressed as $1 + ai + bj$ with $i \in [2, b+1]$. Define

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 1,2 \mod 4 \text{ or } i \text{ is even, } j \equiv 0,3 \mod 4; \\ 1 & \text{if } i \text{ is odd, } j \equiv 0,3 \mod 4 \text{ or } i \text{ is even, } j \equiv 1,2 \mod 4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x, y, z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x, y, z are distinct integers satisfying $ax + by = (a+b)z$. Define i such that $x, y \in S_i$. From Lemmas 5 and 7, it follows that there can be no monochromatic solution if $i \in [-2, M+2]$. From Lemma 6, there is no monochromatic solution in $\bigcup_{i=M+4}^{a+b-4} S_i$. Note that for $i = M+4$, $s_i = 1 + (M+2)a - 2b$. Thus $\chi(S_{M+4}) = 0011$ when M is odd and 1100 when M is even. Similarly for $i = M+5$, $s_i = 1 + (M+3)a - 2b$, and $\chi(S_{M+5}) = 0011$ when M is even and 1100 when M is odd. By Lemma 5, $i \notin \{M+3, M+4\}$. By (1) and Lemma 7, there remain the cases $i \in \{M+5, M+6, a+b-6, a+b-5, a+b-4, a+b-3\}$.

i	{x,y}	$\{\chi(x),\chi(y)\}$	z	χ(z)
M+5	{1+(M+3)a-2b,1+(M+6)a+b}	{0,1}	1+(M+3)a+b	0
M+6	{1+(M+4)a-2b,1+(M+7)a+b}	{1,0}	1+(M+4)a+b	1
M+6	{1+(M+3)a-3b,1+(M+6)a}	{0,0}	1+(M+3)a	1
a+b-6	{1-a+5b,1-4a+2b}	{1,1}	1-a+2b	0
a+b-6	{1-2a+4b,1-5a+b}	{0,1}	1-2a+b	0
a+b-5	{1+5b,1-3a+2b}	{0,0}	1+2b	1
a+b-5	{1-a+4b,1-4a+b}	{1,0}	1-a+b	1
a+b-4	{1+a+5b,1-2a+2b}	{0,1}	1+a+2b	0
a+b-4	{1+4b,1-a+3b}	{0,0}	1+3b	1
a+b-4	{1-2a+2b,1-3a+b}	{1,1}	1-2a+b	0
a+b-3	{1-2a+b,1-a+2b}	{0,0}	1-2a+2b	1
a+b-3	{1+3b,1+a+4b}	{1,1}	1+4b	0

In each case, the solution sets are not monochromatic.

Subcase (ii) If $a \equiv 3 \mod 4$, the smallest possible value of $a+b$ that satisfies this case is 11, with $(a,b) = (3,8)$. Define $\chi : [1, 4(a+b)] \to \{0,1\}$ by

$$\chi(S_i) = \begin{cases} 0011 & \text{if } i \in [-3,2] \text{ and } i \text{ is odd;} \\ 1100 & \text{if } i \in [-3,2] \text{ and } i \text{ is even.} \end{cases}$$

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-3}^{2} S_i$, by Lemma 4, n can be uniquely expressed as $1+ai+bj$ with $i \in [1,b]$. Define

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \text{ is odd, } j \equiv 0,3 \mod 4 \text{ or } i \text{ is even, } j \equiv 1,2 \mod 4; \\ 1 & \text{if } i \text{ is odd, } j \equiv 1,2 \mod 4 \text{ or } i \text{ is even, } j \equiv 0,3 \mod 4. \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x,y,z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x,y,z are distinct integers satisfying $ax+by = (a+b)z$. Define i such that $x,y \in S_i$. Note that $\bigcup_{i=a+b-6}^{a+b-1} S_i$ also satisfies the second colouring for $i \in [4, b+3]$, S_1, S_2 also satisfy the second colouring for $i \in [1,b]$, and S_3 also satisfies the first colouring in this case. From Lemmas 5 and 6, it follows that there can be no monochromatic solution if $i \in [-2,2]$. From Lemma 7, there is no monochromatic solution in $\bigcup_{i=1}^{a+b-1} S_i$. By (1) and Lemma 5, there remain the cases $i \in \{3, a+b-3\}$. But $\chi(S_3) = \chi(S_{-3}) = 0011$. By (1), $r \neq 3$, so that there is no monochromatic solution in this case by Lemma 5.

We now consider the case $a > 2b$. Set $m = \lfloor \frac{a}{b} \rfloor$. For $a > 2b$, the smallest possible value of $a+b$ that satisfies this case is 13, with $(a,b) = (9,4)$. Define $\chi : [1, 4(a+b)] \to \{0,1\}$ by

$$\chi(S_i) = \begin{cases} 0011 & \text{if } i \in [-3,m+1] \text{ and } i \text{ is odd;} \\ 1100 & \text{if } i \in [-3,m+1] \text{ and } i \text{ is even.} \end{cases}$$

For $n \in [1, 4(a+b)] \setminus \bigcup_{i=-3}^{m+1} S_i$, by Lemma 4, n can be uniquely expressed as $1+ai+bj$ with $j \in [3, a+2]$. Define

$$\chi(1+ai+bj) = \begin{cases} 0 & \text{if } i \equiv 1,2 \mod 4, \ j \text{ is even or } i \equiv 0,3 \mod 4, \ j \text{ is odd;} \\ 1 & \text{if } i \equiv 1,2 \mod 4, \ j \text{ is odd or } i \equiv 0,3 \mod 4, \ j \text{ is even.} \end{cases}$$

We show that χ admits no monochromatic solution to $ax+by = (a+b)z$ with x,y,z distinct integers. Suppose $\chi(x) = \chi(y) = \chi(z)$, where x,y,z are distinct integers satisfying $ax+by = (a+b)z$. Define i such that $x,y \in S_i$. Note that $S_{m+2}, S_{m+3}, S_{m+4}$ also satisfy the first colouring, and $\bigcup_{i=-6}^{0} S_i$ also satisfies the second colouring for $j \in [0, a-1]$. From Lemmas 5 and 6, it follows that there can be no monochromatic solution if $i \in [-2, m+3]$. From Lemma 6, there is no monochromatic solution in $\bigcup_{i=-6}^{0} S_i$. By (1) and Lemma 6, there remains the case $i = m+4$. But since $|r| = 1$ in this case, $\{x,y,z\} \in \bigcup_{i=m+2}^{a+b-3} S_i$, which contradicts Lemma 7. ■

The results given by Theorems 2,4,6 and 7 completely determine the Rado number corresponding to the equation $ax+by = (a+b)z$. We state this as our final result.

Theorem 9. *Let a, b be relatively prime positive integers. Then*

$$n(a,b) = \begin{cases} 4(a+b) - 1 & \text{if } a = 1, \ 4 \mid b \text{ or } (a,b) = (3,4); \\ 4(a+b) + 1 & \text{otherwise.} \end{cases}$$

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